Dynamics of the Ablowitz-Ladik soliton train

E. V. Doktorov,[1,*] N. P. Matsuka,[2,†] and V. M. Rothos[3,‡]
[1]*B. I. Stepanov Institute of Physics, 68 F. Skaryna Avenue, 220072 Minsk, Belarus*
[2]*Institute of Mathematics, 11 Surganov Street, 220072 Minsk, Belarus*
[3]*School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, United Kingdom*
(Received 27 February 2004; published 20 May 2004)

It is shown that dynamics of a train of N weakly interacting Ablowitz-Ladik solitons with (almost) equal velocities and masses is governed by the complex Toda chain model. The integrability of the complex Toda chain model provides the means to describe analytically various dynamical regimes of the N-soliton train and to predict initial soliton parameters responsible for each of the regimes. Numerical simulations corroborate well analytical predictions. A specific feature arising for the discrete soliton train system is the appearance of an additional (with respect to the lattice spacing) spatial scale-intersoliton distance. We comment on interplay between both spatial scales.

DOI: 10.1103/PhysRevE.69.056607 PACS number(s): 42.65.Tg, 02.30.Ik, 05.45.Yv

I. INTRODUCTION

Currently energy localization in nonlinear lattices has attracted a great deal of attention. For recent reviews we refer to Refs. [1,2]. Two equations are frequently used to study nonlinear discrete systems—the discrete nonlinear Schrödinger (DNLS) equation and the Ablowitz-Ladik (AL) equation. Though both of them represent discretizations of the completely integrable continuous nonlinear Schrödinger (NLS) equation, their properties are crucially different. The DNLS equation, arising in many areas as an adequate model, is nonintegrable [3,4] and admits quantitative investigation numerically, as a rule. Contrary, it is the AL equation that makes the integrable discrete counterpart of the NLS equation [5], hence enabling us to perform as complete as possible analytical description of an underlying discrete system.

As compared with the DNLS equation, the AL equation found a rather limited area of applications. Nevertheless, there exist at least two reasons justifying a comprehensive analysis of the AL equation. First, this equation serves as a proving ground for testing new analytical and numerical approaches to study discrete systems (e.g., study of numerical homoclinic instabilities for the DNLS and AL equations [6]). Second, for a definite region of parameters, the DNLS equation can be treated as a perturbed version of the AL equation, thereby allowing for considerable progress in the analytical study in the framework of the AL soliton perturbation theory. This idea has been initiated in early papers by Vakhnenko and Gaididei [7] on a soliton motion in a discrete molecular chain, by Kivshar and Campbell [8] on a modeling of the Peierls-Nabarro potential barrier, and by Aceves *et al.* [9] on the self-trapping phenomenon in one-dimensional waveguide arrays [10]. Recently, the same approach has been employed to study discrete soliton dynamics in random media [11], energy transport in α-helical proteins [12], and dissipative coherent structures in discrete systems [13,14]. A model of two coupled AL lattices which admits a reduction to integrable symmetric states was considered by Malomed and Yang [15].

In this paper we address a new issue concerning the discrete solitons—to describe analytically dynamics of a discrete soliton train. An interesting problem of characterization of solitons present in numerical or experimental data has been discussed by Boiti *et al.* [16] in the context of a train of envelope wave pulses modeled by the AL equation on a finite interval. By a soliton train is meant an ordered sequence of N discrete (temporal or spatial) weakly interacting solitons propagating (or extending) in the same direction with (almost) equal velocities. Though this problem has been considered for continuous systems [17–20], a novel feature arises in the discrete case. Namely, two spatial scales appear in a discrete system for a soliton train-lattice spacing and intersoliton distance. Hence, a question can be posed about interplay between these two scales. Recently Soto-Crespo *et al.* [21] put forward some arguments in favor of a relevance of the AL equation for the description of soliton dynamics in a waveguide array with account for coupling through a nonlinear medium located between the waveguides. In this connection, as an example of a concretization of the problem posed in our paper, we can refer to simultaneous launching of power to different places of a nonlinear waveguide array. Evidently, different distributions of power are possible at the exit of this device. We give for the AL equation an efficient analytical description, on the basis of the complex Toda chain (CTC) model, of various regimes of power distribution and, in particular, predict values of initial soliton parameters to provide a bound-state-like evolution of the discrete soliton train.

In Sec. I we formulate a model and identify a region of the AL solitons' parameters which provide a trainlike configuration of solitons. Section II is devoted to a derivation of the Toda chain model. Here we emphasize a crucial difference between the dynamical regimes admitted by the real and complex Toda models. In Sec. III we continue a discussion of the interrelation between the AL model and the CTC model and perform a detailed analysis of such an interrela-

*Electronic address: doktorov@dragon.bas-net.by
†Electronic address: matsuka@im.bas-net.by
‡Electronic address: v.m.rothos@qmul.ac.uk

tion for the particular case of $N=3$. From an analysis of the integrals of motion of the CTC model we predict values of initial AL solitons' parameters responsible for particular dynamical regimes of the AL soliton train dynamics. Section IV contains a comparison of analytical predictions with the results of numerical simulations.

II. THE AL SOLITON TRAIN MODEL

As is well known [5], the AL equation (the overdot denotes time derivative)

$$i\dot{u}_n + \frac{1}{h^2}(u_{n+1} + u_{n-1} - 2u_n) + |u_n|^2(u_{n+1} + u_{n-1}) = 0 \tag{2.1}$$

for a scalar complex function u defined on an infinite 1D lattice with the lattice spacing h has an exact soliton solution

$$u_n(t) = \frac{\sinh h\mu}{h}\frac{\exp[ikh(n-x) + i\alpha]}{\cosh h\mu(n-x)}, \tag{2.2}$$

where

$$x(t) = \frac{2t}{h}\frac{\sinh h\mu}{h^2\mu}\sin hk + x^{(0)},$$

$$\alpha(t) = \frac{2t}{h^2}\left(\cosh h\mu \cos hk + \frac{k}{\mu}\sinh h\mu \sin hk - 1\right) + \alpha^{(0)}.$$

The solution (2.2) depends on four real parameters μ (soliton mass $M=2h\mu$), k [group velocity $v_{gr}=(2/h^2\mu)\sinh h\mu \sin hk$ and phase velocity $v_{ph}=(2/h^2k)(1-\cosh h\mu \cos hk)$], $x^{(0)}$ (soliton initial position), and $\alpha^{(0)}$ (initial phase). The inverse spectral transform method provides the complete analytical description of the AL soliton interaction in a generic case of N solitons moving with pairwise different velocities and being asymptotically free for $n\to\pm\infty$ [5]. Contrary, we are interested here in a chain-like configuration of an ordered sequence of N ($N\geqslant 2$) weakly interacting solitons with (nearly) equal masses and velocities that are spaced apart almost equally. In other words, we suppose that the N-soliton solution to the AL equation for the soliton train is a result of the evolution of the initial configuration

$$u_n(0) = \sum_{j=1}^{N} u_n^{(j)}(0),$$

representing a sum of weakly overlapping one-soliton excitations each of which is characterized by the parameters μ_j, k_j, $x_j^{(0)}$, and $\alpha_j^{(0)}$. The smallness of the adjacent soliton overlapping is determined by a small parameter

$$\epsilon = \exp(-h\mu|x_j^{(0)} - x_{j\pm1}^{(0)}|), \tag{2.3}$$

$\epsilon\ll 1$ for all j and μ is the mean value $\mu=N^{-1}\Sigma_{j=1}^{N}\mu_j$. We consider the interaction force between the neighboring solitons being of the order of their overlap. Therefore, we restrict ourselves by the nearest-neighbor interaction.

Now we formulate more precisely conditions on the solitons' parameters providing the trainlike configuration. Let us denote $y_j=h\mu_j(n-x_j)$ and take $x_{j+1}>x_j$. Then

$$y_{j+1} = \left(1 - \frac{\mu_{j+1}-\mu_j}{\mu_j}\right)y_j - h\mu_j(x_{j+1}-x_j).$$

We pose

$$|\mu_{j\pm1} - \mu_j| \ll \mu. \tag{2.4}$$

This condition means that solitons have almost equal masses. Therefore, $y_{j+1}-y_j\approx -h\mu(x_{j+1}-x_j)$ and in accordance with our assumption about the tail-tail interaction we should pose

$$h\mu|x_{j\pm1} - x_j| \gg 1. \tag{2.5}$$

Considering the phase difference $\chi_{j+1,j}$ of the phases $hk_j(n-x_j)+\alpha_j$ of adjacent solitons, we represent it as

$$\chi_{j+1,j} = \mu^{-1}(k_{j+1}-k_j)y_j - hk_{j+1}(x_{j+1}-x_j) + \alpha_{j+1} - \alpha_j.$$

The proximity of soliton velocities implies the validity of the condition

$$|k_{j\pm1} - k_j| \ll \mu, \tag{2.6}$$

with the result that the phase difference is written as

$$\chi_{j+1,j} = -hk_{j+1}\left(1 + \frac{\mu-\mu_{j+1}}{\mu_{j+1}}\right)(x_{j+1}-x_j) + \alpha_{j+1} - \alpha_j.$$

Finally, the condition

$$h|\mu_j - \mu||x_j - x_{j\pm1}| \ll 1 \tag{2.7}$$

makes it possible to write the phase difference as

$$\chi_{j+1,j} = -hk(x_{j+1}-x_j) + \alpha_{j+1} - \alpha_j$$

with the mean value $k=N^{-1}\Sigma_{j=1}^{N}k_j$. The conditions (2.4)–(2.7) represent the discrete analog of the corresponding conditions formulated within the quasiparticle approach for continuous equations [22].

III. THE COMPLEX TODA CHAIN MODEL

Substituting the sumlike solution into Eq. (2.1), we obtain that the jth soliton is governed by the perturbed AL equation

$$i\dot{u}_n^{(j)} + \frac{1}{h^2}(u_{n+1}^{(j)} + u_{n-1}^{(j)} - 2u_n^{(j)}) + |u_n^{(j)}|^2(u_{n+1}^{(j)} + u_{n-1}^{(j)}) = \epsilon r_n^{(j)}, \tag{3.1}$$

where the perturbation (the overbar means complex conjugation)

$$\epsilon r_n^{(j)} = -|u_n^{(j)}|^2(u_{n+1}^{(j-1)} + u_{n-1}^{(j-1)} + u_{n+1}^{(j+1)} + u_{n-1}^{(j+1)}) - [u_n^{(j)}(\overline{u}_n^{(j-1)} + \overline{u}_n^{(j+1)}) + \overline{u}_n^{(j)}(u_n^{(j-1)} + u_n^{(j+1)})](u_{n+1}^{(j)} + u_{n-1}^{(j)}) \tag{3.2}$$

results from the interaction of neighboring solitons with the jth soliton. Within the limits of the adiabatic approximation of the soliton perturbation theory, a perturbation-induced evolution of the parameters μ_j and k_j is given by the equations [7]

$$\dot{\mu}_j = \frac{\sinh h\mu_j}{h} \sum_{n=-\infty}^{\infty} \frac{\mathrm{Im}(\epsilon R_n^{(j)})\cosh y_j}{\cosh(y_j + h\mu_j)\cosh(y_j - h\mu_j)},$$

$$\dot{k}_j = -\frac{\sinh h\mu_j}{h} \sum_{n=-\infty}^{\infty} \frac{\mathrm{Re}(\epsilon R_n^{(j)})\sinh y_j}{\cosh(y_j + h\mu_j)\cosh(y_j - h\mu_j)}, \tag{3.3}$$

where $R_n^{(j)} = r_n^{(j)}\exp[-ihk_j(n-x_j) - i\alpha_j]$. Substituting the soliton (2.2) into the perturbation formula (3.2) and performing calculation according to Eq. (3.3), we obtain evolution of μ_j and k_j in the form (no summation in repeated indices)

$$\dot{\mu}_j = 8\frac{\sinh^3 h\mu}{h^4} \sum_{l=j\pm 1} s_{jl} e^{-\tilde{\Delta}_{jl}} [\cos \tilde{\chi}_{jl}(\coth h\mu - (h\mu)^{-1})\sin hk + \sin \tilde{\chi}_{jl} \cos hk], \tag{3.4a}$$

$$\dot{k}_j = 8\frac{\sinh^3 h\mu}{h^4} \sum_{l=j\pm 1} s_{jl} e^{-\tilde{\Delta}_{jl}} [\sin \tilde{\chi}_{jl}(\coth h\mu - (h\mu)^{-1})\sin hk - \cos \tilde{\chi}_{jl} \cos hk]. \tag{3.4b}$$

Here $s_{j,j\pm 1} = \mp 1$, $\tilde{\Delta}_{jl} = s_{jl}\Delta_{jl}$, $\tilde{\chi}_{jl} = s_{jl}\chi_{jl}$, and

$$\Delta_{jl} = h\mu(x_j - x_l), \quad \chi_{jl} = -hk(x_j - x_l) + \alpha_j - \alpha_l. \tag{3.5}$$

In the process of calculating the sums in Eq. (3.3) we invoked the Poisson summation formula [23]

$$\sum_{n=-\infty}^{\infty} f(n\mu) = \frac{1}{\mu}\int_{-\infty}^{\infty} \mathrm{d}y\, f(y)\left[1 + 2\sum_{s=1}^{\infty} \cos\frac{2\pi sy}{\mu}\right]$$

and neglected the terms with the factor $\epsilon \exp(-\pi^2 s/\mu), s = 1, 2, \ldots$, which for $\mu \approx 1$ are small. It can be shown that the mean values μ and k calculated along these lines do not depend on time, as should be. As regards the evolution of the parameters x_j and α_j, it is sufficient, within the approximation we adopted, to consider the main contributions

$$\dot{x}_j = \frac{2}{h}\frac{\sinh h\mu_j}{h^2\mu_j}\sin hk_j,$$

$$\dot{\alpha}_j = \frac{2}{h^2}\left(\cosh h\mu_j \cos hk_j + \frac{k_j}{\mu_j}\sinh h\mu_j \sin hk_j - 1\right). \tag{3.6}$$

It follows from Eqs. (3.4) that

$$\dot{\mu}_j + i\dot{k}_j = \frac{8\rho}{h^4} e^{-i\psi} \sinh^3 h\mu \sum_{l=j\pm 1} s_{jl}\exp[s_{jl}(-\Delta_{jl} + i\chi_{jl})],$$

where time-independent quantities ρ and ψ are defined by

$$\rho^2 = [\coth h\mu - (h\mu)^{-1}]^2 \sin^2 hk + \cos^2 hk,$$

$$\psi = \arctan\frac{\cot hk}{\coth h\mu - (h\mu)^{-1}}.$$

Hence, a quantity $\lambda_j = \mu_j + ik_j - (\mu + ik)$ obeys the evolution equation

$$\dot{\lambda}_j = 2\frac{\sinh h\mu}{h^2}(E_{j,j-1} - E_{j+1,j})\rho e^{i\psi}, \tag{3.7}$$

where

$$E_{j,j-1} = (4/h^2)\exp(-\Delta_{j,j-1} + i\chi_{j,j-1} - 2i\psi). \tag{3.8}$$

In turn, it follows from Eqs. (3.5) and (3.6) that

$$-\dot{\Delta}_{jl} + i\dot{\chi}_{jl} = -\frac{2}{h}(\lambda_j - \lambda_l)\rho e^{-i\psi}\sinh h\mu.$$

Therefore,

$$\dot{E}_{j,j-1} = -(2/h)(\lambda_j - \lambda_{j-1})E_{j,j-1}\rho e^{-i\psi}\sinh h\mu.$$

This relation suggests that we can write $E_{j,j-1}$ as $E_{j,j-1} = \exp(q_j - q_{j-1})$, where a complex function q_j obeys the evolution equation

$$\dot{q}_j = -(2/h)\lambda_j \rho e^{-i\psi}\sinh h\mu \tag{3.9}$$

and hence

$$q_j = -h(\mu + ik)x_j + i\alpha_j + (\mu v_{\mathrm{gr}} + ikv_{\mathrm{ph}})t + 2j\ln\frac{2\sinh h\mu}{h} - 2ij\psi, \tag{3.10}$$

with v_{gr} and v_{ph} being the mean group and phase velocities which are given by the same relations as for the single soliton (2.2) but for mean values of μ and k. Now it is straightforward to show due to Eqs. (3.7) and (3.9) that the function q_j obeys the completely integrable complex Toda chain (CTC) equation with N nodes

$$\frac{d^2 q_j}{d\tau^2} = e^{q_{j+1}-q_j} - e^{q_j - q_{j-1}}, \tag{3.11}$$

where $\tau = (2\rho/h^{3/2})t\sinh h\mu$ is the normalized time and we take formally $e^{-q_0} = e^{q_{N+1}} = 0$. The CTC model has been previously derived for the continuous equations [17–20]. Hence, our result is a new evidence of universality of the CTC model for trainlike configurations.

We ought to stress the importance of the fact that Eq. (3.11) describes the *complex* Toda chain model. At first sight, the appearance of exponential interaction in Eq. (3.11) seems quite natural, since it comes from the tail-tail interaction which is essentially linear, governed by the Green function and should be exponential. This simple argument, however, would be valid in the case of the *real* Toda chain (RTC). Second, and it is much more important, the real and complex Toda chain models differ qualitatively from the point of view of dynamics. It is well known [24,25] that the RTC with N nodes admits the Lax representation $(dL/d\tau) = [L, A]$ with $N \times N$ matrices

$$L=\sum_{j=1}^{N}[b_je_{jj}+a_j(e_{j,j+1}+e_{j+1,j})],$$

$$A=\sum_{j=1}^{N}a_j(e_{j+1,j}-e_{j,j+1}),\quad (e_{ij})_{kl}=\delta_{ik}\delta_{jl},$$

$$a_j=\frac{1}{2}\exp\left[\frac{1}{2}(q_{j+1}-q_j)\right],\quad b_j=-\frac{1}{2}\frac{dq_j}{d\tau},\qquad(3.12)$$

$(e_{ij})_{kl}=0$ whenever k or l becomes 0 or $N+1$. In virtue of the algebraic nature of the Lax representation, it is valid for the CTC as well [in this case $b_j=(\sqrt{h}/2)\exp(-i\psi)\lambda_j$]. The Toda model has N integrals of motion in involution, real for the real model [25] and complex for the complex one [18], these integrals being the eigenvalues ζ_j of the matrix L.

For the RTC we have $\zeta_j\neq\zeta_k$ for $j\neq k$. Since $2\zeta_j$ determines the asymptotic velocity of the jth RTC "particle," we conclude that all RTC particles have pairwise different velocities and, as a result, we arrive at a collection of asymptotically free RTC particles.

A completely different situation arises for the CTC. Asymptotic velocities are given by $2\xi_j\equiv 2\,\mathrm{Re}\,\zeta_j$, $j=1,\ldots,N$, and we can have $2\,\mathrm{Re}\,\zeta_j=2\,\mathrm{Re}\,\zeta_k$ together with $\zeta_j\neq\zeta_k$. As a result, we can generally ($\zeta_j\neq\zeta_k$ for $j\neq k$) discriminate between the asymptotically free regime (AFR) when $\xi_j\neq\xi_k$ for $j\neq k$, the bound state regime (BSR) when $\xi_1=\xi_2=\cdots=\xi_N$, and intermediate regime (IR) when only several of the parameters ξ_j are equal.

There are also some singular and degenerate regimes which correspond to a nongeneric case of coincidence of some eigenvalues ζ_j. This subject is out of the scope of the present paper.

IV. THE AL SOLITON TRAIN VIA THE CTC

It is important that since q_j (3.10) is expressed in terms of position and phase of the jth soliton, we can analyze soliton train dynamics in terms of the CTC. Because ζ_j are integrals of motion, it is sufficient to know their initial values. On the other hand, ζ_j are expressed through soliton parameters. Therefore, we can pose a question: how to specify the set of initial AL soliton parameters for which the N-soliton train will evolve to a prescribed dynamical regime. In particular, to describe the BSR, we should solve the characteristic equation $\det(L-\zeta\mathbb{1})=0$ and impose the restriction $\mathrm{Re}\,\zeta_1=\mathrm{Re}\,\zeta_2=\cdots=\mathrm{Re}\,\zeta_N$.

Let us illustrate this approach on an example of the simplest nontrivial case $N=3$ when a soliton has neighbors both from the left and from the right. For simplicity we consider zero initial velocities $k_j(0)=0$. Our analysis is not comprehensive but we comment on main features of the integrals' arrangement. In accordance with Eq. (3.12),

$$a_j=-\frac{i}{h}\exp\left[-\frac{h\mu}{2}r_0+\frac{i}{2}(\alpha_{j+1}^{(0)}-\alpha_j^{(0)})\right]\sinh h\mu,$$

$$j=1,2;\quad b_j=\frac{\sqrt{h}}{2i}\Delta\mu_j,\quad j=1,2,3,$$

where $\Delta\mu_j=\mu_j-\mu$, $\sum_j\Delta\mu_j=0$ in virtue of $\mathrm{tr}L=0$ and we have taken into account that $\psi=\pi/2$ and $\rho=1$ for $k=0$. Here $r_0=x_2^{(0)}-x_1^{(0)}=x_3^{(0)}-x_2^{(0)}$ denotes the initial intersoliton distance and we take r_0 to be an integer, $r_0=m$. Then $\epsilon=\exp(-\mu mh)$. The characteristic equation for determining ζ_j has the form $\zeta^3+p\zeta+q=0$, where

$$p=\frac{e^{-\mu mh}}{h^2}(e^{i\Gamma_1}+e^{i\Gamma_2})\sinh^2 h\mu-\frac{h}{4}(\Delta\mu_1\Delta\mu_2+\Delta\mu_1\Delta\mu_3+\Delta\mu_2\Delta\mu_3),\qquad(4.1a)$$

$$q=-\frac{\sqrt{h}}{2i}\left[\frac{e^{-\mu mh}}{h^2}(\Delta\mu_3e^{i\Gamma_1}+\Delta\mu_1e^{i\Gamma_2})\sinh^2 h\mu-\frac{h}{4}\Delta\mu_1\Delta\mu_2\Delta\mu_3\right],\qquad(4.1b)$$

and $\Gamma_k=\alpha_{k+1}^{(0)}-\alpha_k^{(0)}$. We will consider two configurations of the soliton masses

$$M1:\quad \Delta\mu_1=\beta,\quad \Delta\mu_2=0,\quad \Delta\mu_3=-\beta,$$

$$M2:\quad \Delta\mu_1=\Delta\mu_3=\gamma,\quad \Delta\mu_2=-2\gamma,$$

β and γ are real-valued parameters. The roots of the third-order algebraic equation are given by the well-known Cardano formulas

$$\zeta_1=A+B,\quad \zeta_2=\omega A+\omega^2B,\quad \zeta_3=\omega^2A+\omega B,$$

where

$$A=\sqrt[3]{-\frac{q}{2}+\sqrt{Q}},\quad B=\sqrt[3]{-\frac{q}{2}-\sqrt{Q}},$$

$$Q=\frac{q^2}{4}+\frac{p^3}{27},\quad \omega=\exp\left(\frac{2\pi i}{3}\right).\qquad(4.2)$$

Their analysis make it possible to discriminate between different possibilities of the roots' structure.

A. The case of real *p* and *q*

(i) $Q<0$ leads to $p<p_{cr}=-3(q^2/4)^{1/2}$ and all roots become real and generically pairwise different. Hence we obtain the AFR. For a special choice of the parameters the AFR is transformed to the IR.

(ii) $Q>0$ and $q\neq0$. This leads to real A and B. Hence one root is real and the other two are complex conjugate. This situation corresponds to the IR.

(iii) $Q>0$ and $q=0$. The characteristic equation gives $\zeta_1=0$, $\zeta_{2,3}=\pm\sqrt{-p}$, $p>0$. Hence, all roots have $\mathrm{Re}\,\zeta_j=0$ and we arrive at the BSR.

(iv) $Q=0$. All the roots are real which means the AFR.

B. The case of $p=\overline{p}$ and $q=-\overline{q}$

(i) $Q>0$. This gives $A=-\overline{B}$ and, as a result, $\zeta_j=-\overline{\zeta_j}$. We arrive at the BSR.

(ii) $Q<0$. Then $\zeta_1=-\overline{\zeta_1}$ and $\zeta_3=-\overline{\zeta_2}$ which leads to the AFR.

Let us consider in more detail the most interesting regime, the BSR. We have predicted above two possibilities to choose the root arrangement for producing the BSR:

$$\text{(a)} \quad p>0, \quad q=0,$$

$$\text{(b)} \quad p=\overline{p}, \quad q=-\overline{q}, \quad \frac{q^2}{4}+\frac{p^3}{27}>0.$$

For case (a) we should pose $\Gamma_1=-\Gamma_2=\Upsilon$ in Eq. (4.1) to provide reality of p which gives for the configuration $M1$

$$p_{M1}=\frac{1}{2}\nu^2\cos\Upsilon+\frac{1}{4}h\beta^2, \quad q_{M1}=\frac{1}{4}\nu^2 h\beta\sin\Upsilon,$$

and for the configuration $M2$

$$p_{M2}=\frac{1}{2}\nu^2\cos\Upsilon+\frac{3}{4}h\gamma^2, \quad q_{M2}=i\gamma\frac{\sqrt{h}}{4}(\nu^2\cos\Upsilon+h\gamma^2).$$

Here $\nu=(2/h)\exp(-(1/2)\mu mh)\sinh h\mu$. The condition $q_{M1}=0$ gives $\Upsilon=n\pi$, $n=0,1$ and $p_{M1}=h\beta^2/4+(-1)^n\nu^2/2$. Hence, $p_{M1}>0$ for $n=0$, while for $n=1$ $p_{M1}>0$ if

$$|\beta|>(2/h)^{1/2}\nu. \tag{4.3}$$

Similarly, $q_{M2}=0$ gives $\cos\Upsilon=-h\gamma^2/\nu^2$ and $p_{M2}=h\gamma^2/4>0$, together with the restriction $h\gamma^2/\nu^2<1$. Case (b) survives for the configuration $M2$ only and leads to the condition for the BSR of the form $-(\pi/2)<\Upsilon<(\pi/2)$ and $(9/8)h\gamma^2/\nu^2>1$.

Now we consider some features of the case (a) with $\Upsilon=\pi$ and restrict ourselves to the symmetric solution of the $N=3$ CTC model [19],

$$q_1(\tau)=\ln\frac{\cosh(2\zeta_1\tau+\delta)+1}{4\zeta_1^2}, \quad q_2=0, \quad q_3=-q_1, \tag{4.4}$$

δ is a complex constant. Here the eigenvalues are given by $\zeta_1=i\sqrt{p}$, $\zeta_2=0$, $\zeta_3=-i\sqrt{p}$, $p=(1/4)h\beta^2-(1/2)\nu^2>0$. A useful characteristic illustrating the BSR of the soliton train is the distance between the extreme left and right solitons, in our case $2r(\tau)=x_3(\tau)-x_1(\tau)$. It can be shown from Eq. (4.4) that

$$r(\tau)=\frac{1}{h\mu}\ln\left[\frac{\sinh^2 h\mu}{h^2 p}[\cosh\,\delta_r+\cos(2\sqrt{p}\tau+\delta_i)]\right]. \tag{4.5}$$

Here $\delta=\delta_r+i\delta_i=\ln(w_3/w_1)$ and w_j is a first component of the eigenvector v_j, $(L-\zeta_j)v_j=0$, normalized by $(v_j,v_j)=1$. Hence, the extremal distances between solitons are given by

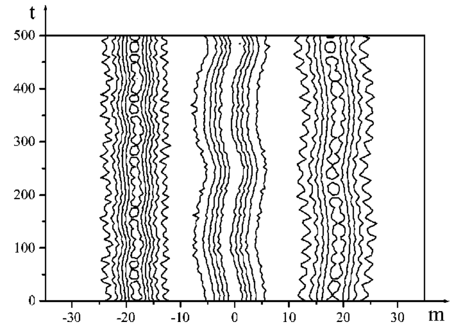

FIG. 1. Quasiequidistant propagation of the three-soliton train provided by numerical solution of the AL equation with the predicted parameters $h=0.9$, $\mu=0.5$, $m=18$, and $\beta=0.05$.

$$r\binom{\max}{\min}=\frac{1}{h\mu}\ln\left[\frac{\sinh^2 h\mu}{h^2 p}(\cosh\,\delta_r\pm 1)\right].$$

Solving the eigenvalue problem, we obtain

$$\delta_r=\ln\frac{\sigma+\sqrt{\sigma^2-1}}{\sigma-\sqrt{\sigma^2-1}}, \quad \delta_i=\pi, \quad \sigma=\sqrt{\frac{h}{2}}\frac{\beta}{\nu}>1. \tag{4.6}$$

Hence, for solitons with $\Upsilon=\pi$, that is with alternating phases $(0,\pi,0)$ and the mass configuration $M1$, the minimal distance coincides with the initial one, $r_{\min}=r_0$, while the maximal distance is equal to

$$r_{\max}=r_0+\frac{1}{h\mu}\ln\frac{\sigma^2}{\sigma^2-1}.$$

As it follows from Eq. (4.5), a period T of oscillations between the maximal and minimal distances is given by $T=\pi h^{3/2}/2p^{1/2}\sinh h\mu$. It should be stressed that the choice of the initial soliton parameters is rather flexible and we can vary to some extent the intersoliton distance and lattice spacing maintaining the smallness of $\epsilon=\exp(-\mu mh)$.

V. COMPARISON WITH NUMERICAL SIMULATIONS

In this section we compare the predictions of the CTC model with the results of numerical solution of the AL equation. At first we consider the BSR. It follows from the preceding Section that for the three-soliton train with alternating

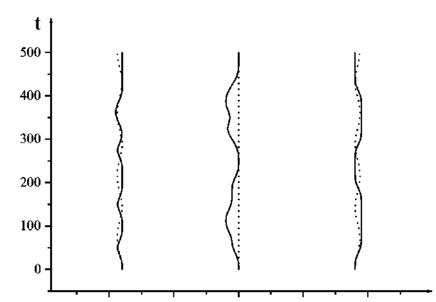

FIG. 2. Trajectories of solitons within the three-soliton train. Solid curves give numerical results, dashed curves correspond to predictions from the symmetric solution of the Toda chain model. Parameters are the same as in Fig. 1.

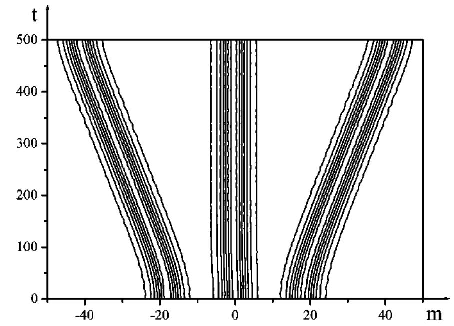

FIG. 3. The appearance of the asymptotically free regime when the condition (4.3) is not fulfilled. The parameters are the same as in Fig. 1, except for $\beta=0.01$.

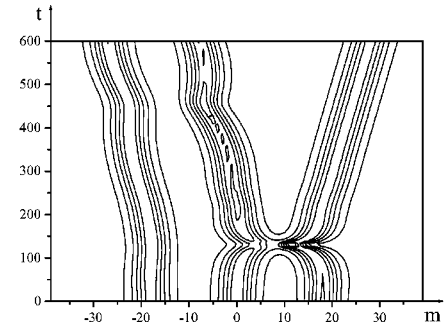

FIG. 4. The asymptotic intermediate regime of the three-soliton train propagation corresponding to the choice of soliton parameters (ii) in Sec. IV A. The phase arrangement is $(2\pi/3,0,-2\pi/3)$, the parameters h, μ, and m are the same as before, the parameter β is calculated by the formula given in the text. The initial stage of the train evolution demonstrates a transient process in the soliton interaction.

phases $(0,\pi,0)$ and mass configuration $M1$ we arrive at the BSR provided the condition (4.3) is fulfilled. Let us choose the parameters as $h=0.9$ and $\mu=0.5$ with $r_0=m=18$ which gives $(2/h)^{1/2}\nu=0.017$. Then $\beta=0.05$ obeys the condition (4.3). Such a choice of the parameters corresponds to the same value of ϵ (2.3) that was used for simulations in the continuous models [18,19]. Figure 1 illustrates the numerical solution of the AL equation for the three-soliton train with the above parameters. We see that the Toda chain model predicts correctly the quasiequidistant regime of the train dynamics for the large time interval (up to $t=500$ at least). A comparison of the soliton trajectories $x_j(t)$ calculated from Eqs. (3.10) and (4.4) and directly by solving the AL equation is given in Fig. 2.

To illustrate the AFR, it is sufficient to note that a violation of the condition (4.3) leads to appearance of pairwise different real parts of the eigenvalues ζ_j and hence to the AFR. Hence, for $\beta=0.01$ we predict the AFR. This prediction is confirmed numerically (see Fig. 3).

Finally, the IR is realized asymptotically for choice (ii) in Sec. IV A. Taking for definiteness $p=0$, we obtain for the configuration $M1$ $\cos\Upsilon=-h\beta^2/2\nu^2$. Hence we can choose $\Upsilon=2\pi/3$ which gives $\beta=\nu/\sqrt{h}$. Taking the soliton parameters as $h=0.9$, $\mu=0.5$, and $m=18$ with β calculated by means of the above formula, numerical solution of the AL equation demonstrates, after some transient period, the steady state intermediate regime (Fig. 4).

VI. CONCLUSION

We have shown in this paper that the CTC model provides an adequate basis to describe adiabatic dynamics of the AL N-soliton train. Comparison of analytical predictions with numerical simulations demonstrates a very good agreement. We conjecture that because of the universality of the CTC model for description of N-soliton train dynamics, the above analysis can be implemented to actual discrete physical problems, despite a restricted applicability of the AL model. It should be pointed out in this connection that starting from a nonintegrable discrete soliton model, we will still derive a perturbed CTC model. In the case of a small perturbed term, a method developed by Garnier and Abdullaev [26] can be successfully employed to the perturbed Toda equation. We mention as well that the adiabatic soliton dynamics does not account for radiation which could arise as a result of a soliton interaction within the train [27]. A formalism to incorporate perturbation-induced radiation effects for the AL soliton has been proposed in Ref. [28] in the framework of the Gel'fand-Levitan-like summation equations and in Ref. [29] on the basis of the Riemann-Hilbert problem.

ACKNOWLEDGMENTS

The authors are grateful to V. S. Gerdjikov for helpful discussions. This work was partly supported by the Belarussian Foundation for Fundamental Research, Grant No. Φ01 -052, and by UK-EPSRC, Grant No. GR/S47335/01.

[1] P. G. Kevrekidis, K. Ø. Rasmussen, and A. R. Bishop, Int. J. Mod. Phys. B **15**, 2833 (2001).
[2] J. C. Eilbeck and M. Johansson, in *Localization and Energy Transfer in Nonlinear Systems*, edited by L. Vázquez, R. S. MacKay, and P. M. Zorzano (World Scientific, Singapore, 2003), p. 44.
[3] B. M. Herbst and M. J. Ablowitz, Phys. Rev. Lett. **62**, 2065 (1989).
[4] R. Scharf and A. R. Bishop, Phys. Rev. A **43**, 6535 (1991).
[5] M. J. Ablowitz and J. F. Ladik, J. Math. Phys. **17**, 1011 (1976).
[6] M. J. Ablowitz and B. M. Herbst, SIAM (Soc. Ind. Appl. Math.) J. Appl. Math. **50**, 339 (1990).
[7] A. A. Vakhnenko and Yu. B. Gaididei, Theor. Math. Phys. **68**, 873 (1986).
[8] Yu. S. Kivshar and D. K. Campbell, Phys. Rev. E **48**, 3077 (1993).
[9] A. B. Aceves, C. De Angelis, T. Peschel, R. Muschall, F. Le-

derer, S. Trillo, and S. Wabnitz, Phys. Rev. E **53**, 1172 (1996).
[10] D. N. Christodoulides and R. J. Joseph, Opt. Lett. **13**, 794 (1988).
[11] J. Garnier, Phys. Rev. E **63**, 026608 (2001).
[12] K. Kundu, Phys. Rev. E **61**, 5839 (2000).
[13] F. Kh. Abdullaev, A. A. Abdumalikov, and B. A. Umarov, Phys. Lett. A **305**, 371 (2002).
[14] Ken-ichi Maruno, A. Ankiewicz, and N. Akhmediev, Opt. Commun. **221**, 199 (2003).
[15] B. A. Malomed and J. Yang, Phys. Lett. A **302**, 163 (2002).
[16] M. Boiti, J. Leon, and F. Pempinelli, Phys. Rev. E **54**, 5739 (1996).
[17] J. M. Arnold, Phys. Rev. E **60**, 979 (1999).
[18] V. S. Gerdjikov, D. J. Kaup, I. M. Uzunov, and E. G. Evstatiev, Phys. Rev. Lett. **77**, 3943 (1996); V. S. Gerdjikov, I. M. Uzunov, E. G. Evstatiev, and G. L. Diankov, Phys. Rev. E **55**, 6039 (1997).
[19] V. S. Gerdjikov, E. V. Doktorov, and J. Yang, Phys. Rev. E **64**, 056617 (2001).
[20] V. S. Shchesnovich, Phys. Rev. E **65**, 046614 (2002).
[21] J. M. Soto-Crespo, N. Akhmediev, and A. Ankiewicz, Phys. Lett. A **314**, 126 (2003).
[22] V. I. Karpman and V. V. Solov'ev, Physica D **3**, 487 (1981).
[23] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), p. 466.
[24] M. Toda, *Theory of Nonlinear Lattices* (Springer Verlag, Berlin, 1989).
[25] J. Moser, Lect. Notes Phys. **38**, 467 (1975); Adv. Math. **16**, 197 (1975).
[26] J. Garnier and F. Kh. Abdullaev, Phys. Rev. E **67**, 026609 (2003).
[27] E. A. Kuznetsov, A. V. Mikhailov, and I. A. Shimokhin, Physica D **87**, 201 (1995).
[28] V. V. Konotop, D. Cai, M. Salerno, A. R. Bishop, and N. Grønbech-Jensen, Phys. Rev. E **53**, 6476 (1996).
[29] E. V. Doktorov, N. P. Matsuka, and V. M. Rothos, Phys. Rev. E **68**, 066610 (2003).